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BEHAVIOUR OF BURIED FLEXIBLE CYLINDERS UNDER THE INFLUENCE OF NONUNIFORM HOOP COMPRESSION

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Abstract—General linearised differential equations of equilibrium are derived for a thin circular cylinder under the influence of nonuniform initial stress. The equations are presented for three alternative ring theories where various combinations of bending, extensional, and transverse shearing deformations are considered. The problem of a long flexible tube buried in an elastic continuum prestressed by nonhydrostatic biaxial field stress is studied. The classical eigenvalue problem is formulated to determine the distributions of nonuniform stress resultants which destabilise the tube in its initial circular configuration (neglecting the influence of prebuckling deformations). The simplest ring theory where only bending deformations are considered is found to provide a simple and accurate solution to the problem, and a parametric study is carried out to facilitate the use of the solution.

INTRODUCTION

The elastic stability of buried flexible cylinders has received considerable attention in recent years, and a number of solutions have been developed for the behaviour of these buried structures under the influence of uniform hoop compressions, e.g. Forrestal and Herrmann[1] and Cheney[2]. However, in-situ field stress is often nonhydrostatic, so that nonuniform hoop compressions, bending moments, and shear forces are generated in the buried cylinder. A recent survey by Baikie and Meyerhof[3] highlights the need for an analysis which considers the influence of these nonuniform stress resultants. In this study, a solution to the problem of a flexible circular tube buried in an elastic continuum under the influence of a biaxial stress field is presented.

First, equations of equilibrium are derived for a circular ring (or cylinder under plane strain conditions) under the influence of nonuniform initial stress resultants. The equations are developed for a number of different levels of approximation, so that the influence of bending, extensional, and shearing deformations can be assessed and the simplest possible solution obtained.

A parametric study of the buried cylinder problem is then presented, and the general behaviour investigated with particular reference made to the influence of nonuniformity of initial stress resultants, the elastic ground parameters, the interface condition, and the effect of extension and shear in the cylinder.

1. GENERAL DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR A THIN RING

In this section, the differential equations of equilibrium for a long cylindrical shell deforming under plane strain conditions, Fig. 1, are described. The equations are developed in the Appendix and are similar to those of Herrmann and Armenkas[4], although, in the present work, midsurface extension and rotation terms are adopted as fundamental variables because they reduce the number of terms and the order of differentiation which occur in the resulting equations.

The circular tube of midsurface radius a and thickness t is assumed to have Young's modulus E_l, Poisson's ratio ν_l and shape factor κ, (e.g. [5]) so that the hoop H, flexural D, and transverse shear $\kappa G_l t$ stiffnesses of the shell are defined as

$$H = \frac{E_l t}{(1 - \nu_l^2)}$$
$$D = \frac{E_l t^3}{12(1 - \nu_l^2)} \tag{1}$$
$$\kappa G_l t = \frac{\kappa E_l t}{2(1 + \nu_l)} .$$

Using the hoop force N, bending moment M, and shear force Q resultants, the equations of equilibrium can be expressed as (see the Appendix):

$$\left[N - \frac{M}{a} + H + \frac{D}{a^2}\right]\epsilon + Q\psi_\theta + \left[M - \frac{D}{a^2}\right]\frac{\partial\psi_\theta}{\partial\theta} - F_\epsilon = 0$$
$$\left[N - \frac{M}{a} + \kappa G_l t\right]\alpha + \left[-\frac{M}{a} + \kappa G_l t\right]\psi_\theta - F_\alpha = 0 \tag{2}$$
$$Q\epsilon - \frac{\partial}{\partial\theta}\left[\left(\frac{M}{a} - \frac{D}{a^2}\right)\epsilon\right] + \left[-\frac{M}{a} + \kappa G_l t\right]\alpha$$
$$- \left[\frac{M}{a} - \kappa G_l t\right]\psi_\theta + \frac{\partial}{\partial\theta}\left(\frac{D}{a^2}\frac{\partial\psi_\theta}{\partial\theta}\right) - \frac{m_\theta}{a} = 0,$$

where the radial w and circumferential v deformation of the tube midsurface are used to define the midsurface extension

$$\epsilon = \frac{1}{a}\left(\frac{\partial v}{\partial\theta} + w\right) \tag{3a}$$

and rotation

$$\alpha = \frac{1}{a}\left(\frac{\partial w}{\partial\theta} - v\right) \tag{3b}$$

and the term ψ_θ represents the linear variation of circumferential displacement across the thickness of the tube. The terms F_ϵ, F_α, and m_θ represent the tractions applied to the structure as it deforms, and using the thickness coordinate ζ, these are defined by:

$$-\frac{\partial F_\alpha}{\partial\theta} + F_\epsilon = [f_\zeta(1 + \zeta/a)]_{-t/2}^{t/2}$$
$$-\frac{\partial F_\epsilon}{\partial\theta} - F_\alpha = [f_\theta(1 + \zeta/a)]_{-t/2}^{t/2} \tag{4}$$
$$m_\theta = [f_\theta\zeta(1 + \zeta/a)]_{-t/2}^{t/2},$$

where f_ζ and f_θ are the tractions applied to the tube surface in the radial and circumferential directions, respectively.

For thin rings, the shear deformations are often negligible. If the shear strain at the midsurface is negligible, then

$$\psi_\theta + \alpha = 0$$

so that Q and ψ_θ can be eliminated from the equations of equilibrium, yielding the simpler expressions

$$\left[N - \frac{M}{a} + H + \frac{D}{a^2}\right]\epsilon - \left[\frac{1}{a}\frac{\partial M}{\partial \theta} + \frac{m_\theta}{a}\right]\alpha + \left[-\frac{M}{a} + D\right]\frac{\partial \alpha}{\partial \theta} - F_\epsilon = 0$$
$$\frac{M}{a}\frac{\partial \epsilon}{\partial \theta} - \frac{\partial}{\partial \theta}\left[\frac{D}{a^2}\epsilon\right] + N\alpha - \frac{\partial}{\partial \theta}\left[\frac{D}{a^2}\frac{\partial \alpha}{\partial \theta}\right] + \frac{m_\theta}{a} - F_\alpha = 0 \qquad (5)$$

Thin rings have high membrane stiffness H. Examination of the first equation in (5) indicates that, for small deformations, ϵ is negligible. On this basis, it is reasonable to simplify the second equation of (5) to:

$$N\alpha - \frac{\partial}{\partial \theta}\left[\frac{D}{a^2}\frac{\partial \alpha}{\partial \theta}\right] + \frac{m_\theta}{a} - F_\alpha = 0. \qquad (6)$$

2. STABILITY OF BURIED CYLINDERS

Problem definition

The buried tube, Fig. 1, is assumed to be very long, so that it deforms under conditions of plane strain. The ground that supports the tube is considered to be a single-phase isotropic material such as a soil or rock mass, with an incrementally elastic behaviour characterised by two constants: Young's modulus E_s and Poisson's ratio ν_s.

Before insertion of the tube, the ground is assumed to be prestressed with uniform vertical σ_V and horizontal $\sigma_H = K\sigma_V$ field stresses, which induce stress resultants (see, for example, Höeg[6] and Einstein and Schwartz[7])

$$N = N_0 + N_2 \cos 2\theta$$
$$M = M_2 \cos 2\theta \qquad (7)$$
$$Q = Q_2 \sin 2\theta.$$

Two alternative conditions will be assumed to characterise the ground-structure interaction response at the interface.

(a) A perfectly rough condition, resulting in complete compatibility of radial and circumferential displacements, and full transmission of normal and shear tractions, across the interface.

(b) A perfectly smooth condition, where shear stress is not transmitted between the structure and ground, and where circumferential displacements are not continuous, due to interfacial slip.

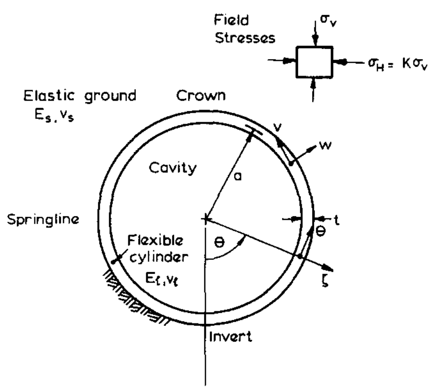

Fig. 1. Coordinate description.

The real interface condition will be somewhere between these two extremes, because there will, in general, be some finite limit to the shear stresses that can be developed between the cylinder and ground.

The tractions applied to the tube by the ground will be assumed to remain constant in direction relative to the initial geometry of the tube, during the critical deformations. The rotation of these interaction forces has been examined in a recent publication of Moore and Booker[8], and was found to be unimportant whenever the supporting ground significantly increases the tube stability.

Ground restraint

A straightforward linear elastic analysis of the elastic continuum surrounding the cylindrical tube, e.g. Timoshenko and Goodier[9], provides a relationship between the harmonic coefficients of the radial $\bar{\sigma}$ and circumferential $\bar{\tau}$ tractions applied to the ground at the interface, and the coefficients of radial $\bar{w}$ and circumferential $\bar{v}$ displacement

$$(\bar{\sigma}, \bar{w}) = (\bar{\sigma}_0, \bar{W}_0) + \sum_{n=2}^{\infty} (\bar{\sigma}_n, \bar{W}_n) \cos n\theta$$
$$(\bar{\tau}, \bar{v}) = \sum_{n=2}^{\infty} (\bar{\tau}_n, \bar{V}_n) \sin n\theta. \tag{8}$$

For harmonic n,

$$\begin{bmatrix} \bar{\sigma}_n \\ \bar{\tau}_n \end{bmatrix} = -\frac{2G_s}{a(3 - 4\nu_s)} \begin{bmatrix} 2n(1 - \nu_s) + 1 - 2\nu_s & n(1 - 2\nu_s) + 2(1 - \nu_s) \\ n(1 - 2\nu_s) + 2(1 - \nu_s) & 2n(1 - \nu_s) + 1 - 2\nu_s \end{bmatrix} \begin{bmatrix} \bar{W}_n \\ \bar{V}_n \end{bmatrix} \tag{9}$$

where the shear modulus of the ground $G_s = E_s/2(1 + \nu_s)$.

Table 1 gives details of the relationship between structural and ground displacements and tractions at the interface, for the perfectly smooth and perfectly rough conditions. Use of these leads to an expression for ground restraint relating the coefficients of radial w and circumferential v displacement and incremental radial σ and circumferential τ tractions on the structure

$$\begin{pmatrix} \sigma_n \\ \tau_n \end{pmatrix} = A^n \begin{pmatrix} W_n \\ V_n \end{pmatrix} \tag{10}$$

where

$$(\sigma, w) = (\sigma_0, W_0) + \sum_{n=2}^{\infty} (\sigma_n, W_n) \cos n\theta$$
$$(\tau, v) = \sum_{n=2}^{\infty} (\tau_n, V_n) \sin n\theta, \tag{11}$$

and A^n is provided in Table 1 for the two ideal interface conditions.

The "tractions" acting on the structure F_ϵ, F_α consist of two components

$$F_\epsilon = \xi^a + \xi^e$$
$$F_\alpha = \eta^a + \eta^e$$

where ξ^a and η^a are the "tractions" which result from the ground restraint at interface, and ξ^e and η^e are any other incremental "tractions" applied to the structure (e.g. as a result of field stresses acting in the elastic continuum).

Table 1. Interface conditions and ground restraint matrix

	Rough interface		Smooth interface	
Stress conditions	$\sigma_n = \bar{\sigma}_n$ $\tau_n = \bar{\tau}_n$		$\sigma_n = \bar{\sigma}_n$ $\tau_n = \bar{\tau}_n = 0$	
Displacement compatability	$W_n = \overline{W}_n$ $V_n = \overline{V}_n$		$W_n = \overline{W}_n$	
Ground restraint matrix	$\dfrac{2n(1-\nu_s)+1-2\nu_s}{(3-4\nu_s)}$	$\dfrac{n(1-2\nu_s)+2(1-\nu_s)}{(3-4\nu_s)}$	$\dfrac{(n^2-1)}{(2n(1-\nu_s)+1-2\nu_s)}$	0
$A^n\left(\dfrac{a}{-2G_s}\right)$	$\dfrac{n(1-2\nu_s)+2(1-\nu_s)}{(3-4\nu_s)}$	$\dfrac{2n(1-\nu_s)+1-2\nu_s}{(3-4\nu_s)}$	0	0

Use of eqns (3, 4) leads to

$$(\xi_n^a \eta_n^a)^T = B^n(\epsilon_n \alpha_n)^T \tag{12a}$$

where

$$B^n = \begin{bmatrix} B_{11}^n & B_{12}^n \\ B_{12}^n & B_{22}^n \end{bmatrix} = \frac{1}{(n^2-1)^2}\begin{bmatrix} A_{11}^n - 2nA_{12}^n + n^2A_{22}^n & n(A_{11}^n + A_{22}^n) - A_{12}^n(n^2+1) \\ n(A_{11}^n + A_{22}^n) - A_{12}^n(n^2+1) & n^2A_{11}^n - 2nA_{12}^n + A_{22}^n \end{bmatrix} \tag{12b}$$

and

$$(F_\epsilon, \epsilon) = (\xi_0^a + \xi_0^e, \epsilon_0) + \sum_{n=2}^{\infty} (\xi_n^a + \xi_n^e, \epsilon_n)\cos n\theta$$
$$(F_\alpha, \alpha) = \sum_{n=2}^{\infty} (\eta_n^a + \eta_n^e, \alpha_n)\sin n\theta. \tag{13}$$

Buried cylinder equations

Substitution of eqns (7, 12) into the differential equation of equilibrium for the inextensional case (6) yields

$$\left\{\frac{N_2}{2}\alpha_4 + \left(N_0 + \frac{4D}{a^2} - B_{22}^2\right)\alpha_2 - \eta_2^e\right\}\sin 2\theta + \left\{\frac{N_2}{2}\alpha_5 + \left(N_0 + \frac{9D}{a^2} - B_{22}^3\right)\alpha_3 - \eta_3^e\right\}\sin 3\theta$$
$$+ \sum_{n=4}^{\infty}\left\{\frac{N_2}{2}(\alpha_{n-2} + \alpha_{n+2}) + \left(N_0 + \frac{n^2D}{a^2} - B_{22}^n\right)\alpha_n - \eta_n^e\right\}\sin n\theta = 0. \tag{14}$$

Because the harmonic functions are orthogonal

$$C^1\boldsymbol{\delta}^1 = \mathbf{r}^1 \tag{15}$$

where

$$\boldsymbol{\delta}^1 = \{\alpha_2, \alpha_3, \alpha_4, \ldots\}$$
$$\mathbf{r}^1 = \{\eta_2^e, \eta_3^e, \eta_4^e, \ldots\}$$

and

$$C^1 = \begin{bmatrix} N_0 + P_2 & 0 & \frac{N_2}{2} & 0 & 0 & 0 & \dots \\ 0 & N_0 + P_3 & 0 & \frac{N_2}{2} & 0 & 0 & \\ \frac{N_2}{2} & 0 & N_0 + P_4 & 0 & \frac{N_2}{2} & 0 & \\ 0 & \frac{N_2}{2} & 0 & N_0 + P_5 & 0 & \frac{N_2}{2} & \\ \vdots & & & & & & \end{bmatrix} \tag{16}$$

with $P_n = n^2D/a^2 - B_{22}^n$. The nonuniform component of hoop force N_2 acts to couple the different coefficients of α, which are solved independently for the uniform hoop force problem ($N_2 = 0$). The superscripts on $\boldsymbol{\delta}$, $\mathbf{r}$, and C refer to the number of degrees of freedom assigned to each harmonic in the series.

If all of the bending, extension, and shear terms are retained, then from (2)

$$C^3\boldsymbol{\delta}^3 = \mathbf{r}^3 \tag{17}$$

where

$$\boldsymbol{\delta}^3 = \{\epsilon_2, \alpha_2, \psi_2, \epsilon_3, \alpha_3, \psi_3, \dots\}^T,$$

$$\mathbf{r}^3 = \{\xi_2^e, \eta_2^e, \frac{m_2}{a}, \xi_3^e, \eta_3^e, \frac{m_3}{a}, \dots\}^T, \tag{18a}$$

$$C^3 = \begin{bmatrix} \chi_2 & 0 & \Phi_2 & 0 & 0 & 0 & \dots \\ 0 & \chi_3 & 0 & \Phi_3 & 0 & 0 & \\ \Phi_2^T & 0 & \chi_4 & 0 & \Phi_4 & 0 & \\ 0 & \Phi_3^T & 0 & \chi_5 & 0 & \Phi_5 & \\ \vdots & & & & & & \end{bmatrix}$$

$$\chi_n = \begin{bmatrix} H + \frac{D}{a^2} + B_{11}^n + N_0 & B_{12}^n & -n\frac{D}{a^2} \\ B_{12}^n & N_0 + \kappa G_l t + B_{22}^n & \kappa G_l t \\ -n\frac{D}{a^2} & \kappa G_l t & \kappa G_l t + \frac{n^2D}{a^2} \end{bmatrix} \tag{18b}$$

$$\Phi_n = \begin{bmatrix} \frac{N_2 - M_2}{2} & 0 & \frac{Q_2}{2} + \frac{(n+2)}{2}M_2 \\ 0 & \frac{N_2 - M_2}{2} & \frac{-M_2}{2} \\ -\frac{Q_2}{2} + \frac{nM_2}{2} & \frac{-M_2}{2} & \frac{-M_2}{2} \end{bmatrix} \tag{18c}$$

with $(\psi_\theta, m_\theta) = \sum_{n=2}^{\infty} (\psi_n, m_n) \sin n\theta$.

Solution of stability problem

The linearised equations of equilibrium (15, 17) relate the harmonic coefficients of deformation $\boldsymbol{\delta}$ with the "tractions" applied to the tube $\mathbf{r}$. The theory has been developed on the basis of small deformations α, ϵ, and ψ_θ (see the Appendix), and the restraint provided by the ground surrounding the tube usually does act to prevent large deformations[10]. The theory can, therefore, be used to determine the prebuckling deformations which occur as a result of the nonhydrostatic field stresses initially present in the elastic continuum (associated with nonzero values of η_2^ξ and ξ_2^ξ). It is also reasonable to neglect the effect of prebuckling deformations and solve for the stress resultants N, M, and Q, which elastically destabilise the tube in its initial circular shape.

To solve the stability problem, the infinite sets of eqns. (15, 17) are truncated, and separated into two components, $C = C_s + \lambda C_N$. The static matrix C_s contains all the terms from C which are independent of the stress level. The stability matrix C_N is a function of the stress resultants N, Q, and M, and is scaled by the stress level factor λ.

The linear eigenvalue problem [$\det(C_s + \lambda_{cr} C_N) = 0$] is then solved for the critical stress level λ_{cr}. A convergence check is necessary to assess the influence of the terms which have been discarded, and a straightforward procedure has been adopted where the number of equations considered is progressively increased until the stress resultants converge to the critical values. The form of the critical deformation can also be determined by obtaining the eigenvector associated with λ_{cr}. The eigenvector will contain many harmonics of α, ϵ, and ψ_θ when the hoop compressions are nonuniform, but for $N_2 = M_2 = Q_2 = 0$ the eigenvector will have one single "critical harmonic."

This solution for the critical nonuniform stress resultants can be simplified to yield less general solutions obtained by other workers. The solution of Smith and Simitses[11] for the uniform hoop force which destabilises an unsupported ring in the "nth" harmonic

$$N_{cr} = -\frac{n^2 D}{n(1 + n^2 D/\kappa G_l t)} \tag{19}$$

is obtained from (17, 18) with $N_2 = B_{11}^\eta = B_{12}^\eta = B_{22}^\eta = 0$. Other solutions for a buried cylinder with uniform hoop stress [1, 8] are then obtained when $\kappa G t$ is very large.

3. PARAMETRIC STUDY

Statement of parametric solution

Consider a long, flexible, cylindrical shell, buried a long way from the surface. Field stresses induce the nonuniform hoop force $N = N_0 + N_2 \cos 2\theta$, which acts to destabilise the tube. The uniform component of critical hoop force N_0 can be written in the form of

$$N_0 = -\frac{3D}{a}(\bar{n}^2 + 1) I R_\nu R_e \tag{20}$$

where $D = E_l t^3/12(1 - \nu_l^2)$ is the flexural rigidity of the cylindrical shell under plane strain conditions; $\bar{n}$ is the approximate buckling harmonic for the uniform hoop compression case[8],

$$\bar{n} = \left(\frac{G_s a^3}{D}\right)^{1/3},$$

I is the influence factor for the effect of nonuniformity of the initial stress resultants in the cylinder, Fig. 2; R_ν is the correction factor for the effect of Poisson's ratio of the soil ν_s and the interface condition, Figs. 3 and 4; and R_e is the correction factor for the effect of extension and shear deformation in the cylinder.

Under most circumstances, the buckling deformations are effectively inextensional with plane sections remaining plane (further discussion appears in a following section)

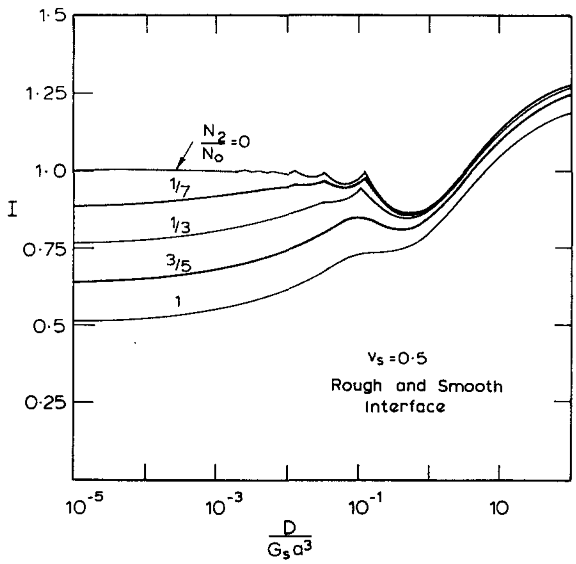

Fig. 2. Influence factor for cylinder stability.

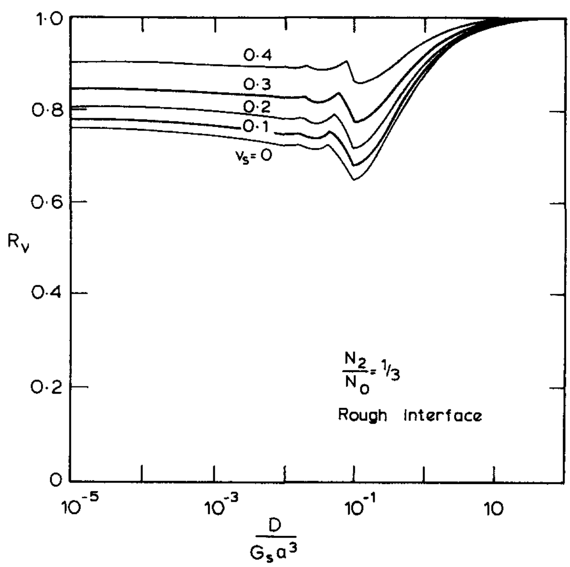

Fig. 3. Correction factor for Poisson's ratio rough interface.

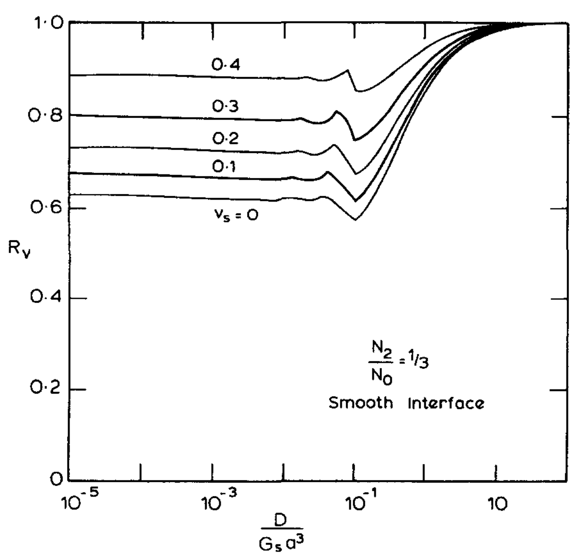

Fig. 4. Correction factor for Poisson's ratio smooth interface.

so that $R_e \simeq 1.0$. Any particular problem is characterised by the level of nonuniformity of hoop compression N_2/N_0, the relative flexural stiffness of the cylinder $D/G_s a^3$, Poisson's ratio of the ground ν_s, and the interface condition, so that the uniform component of critical hoop compression N_0 is obtained using eqn (20) in conjunction with Figs. 2, 3, and 4.

Influence of nonuniformity of hoop compression

Simplified equations (15, 16) have been developed on the basis of $\epsilon = \alpha + \psi_\theta = 0$. These indicate that when there is negligible membrane extension and transverse shear deformation, the structural stability is dependent only on the hoop compressions in the tube. The importance of extension and transverse shear are considered further in another subsection.

Five levels of nonuniformity of hoop compression N_2/N_0 have been presented in the parametric solution, ranging from 0 to 1. The first of these limits represents a perfectly uniform hoop force, and the latter is the combination associated with an unsupported cylinder under uniaxial field stress. This second limit is not achieved in practice, but may be approached in the case of cylinders buried in highly over-consolidated soil.

Figure 2 indicates that the influence coefficient I is strongly influenced by the level of nonuniformity of hoop force N_2/N_0. For soft ground, $D/G_s a^3 > 1$ the influence is slight, but in stiffer ground the effect is quite marked.

As the ground stiffness increases, the stability of the buried tube becomes dependent on the maximum hoop compression rather than both uniform and deviatoric components. In fact, as $D/G_s a^3 \to 0$ the limit of I

$$\lim I = I_u \Big/ \left(1 + \frac{N_2}{N_0}\right) \tag{21}$$

where I_u is the influence coefficient for a uniformly loaded cylinder $N_2/N_0 = 0$. This occurs because the wavelength of the critical deformation (indicated roughly by $2\pi a/\bar{n}$) decreases as the ground stiffens. Eventually, there are one or more buckles in the region of maximum hoop force, so that it is the maximum value $(N_0 + N_2)$ that controls the elastic stability.

Because the limit (21) is also the minimum value, a simple and conservative approach to problems involving nonuniform hoop compressions is to use the maximum hoop compression $(N_0 + N_2)$ as the representative value, and to affect a comparison with the critical hoop compression resulting from an analysis of the uniform problem (e.g. Forrestal and Herrmann[1] and Moore and Booker[8]). Use of the parametric solution by means of (20), with Figs. 2 to 4, is, however, straightforward, and will ensure a more accurate and economical estimate of the critical hoop force. It is important to note that the practice of using the average hoop compression N_0 as a representative value (e.g. Duns and Butterfield[12] and Gumbel and Wilson[13]) will lead to an unconservative solution. This practice was based on the work of Anderson and Boresi[14], and it is not appropriate to elastically supported cylinders.

The nonuniform component of hoop force has another important effect on the critical response of the buried cylinder. When the hoop compression is uniform, it is well known that one isolated mode of deformation is adopted around the complete ring. As the relative stiffness of the ground changes, different modes are preferred, and the critical deformation swaps from one to another, producing the familiar cusped curve shown in Fig. 2 ($N_2/N_0 = 0$). The nonuniformity of hoop compression acts to couple different modes, and there is a more gradual transition from mode to mode. For the extreme case shown, $N_2/N_0 = 1$, the curve for influence coefficient is a smooth one, with no discernable discontinuities in slope.

Influence of interface condition and Poisson's ratio ν_s

The basic theory has been developed for the two alternative types of interface behaviour—perfectly rough and perfectly smooth. Static analysis of the system (e.g.

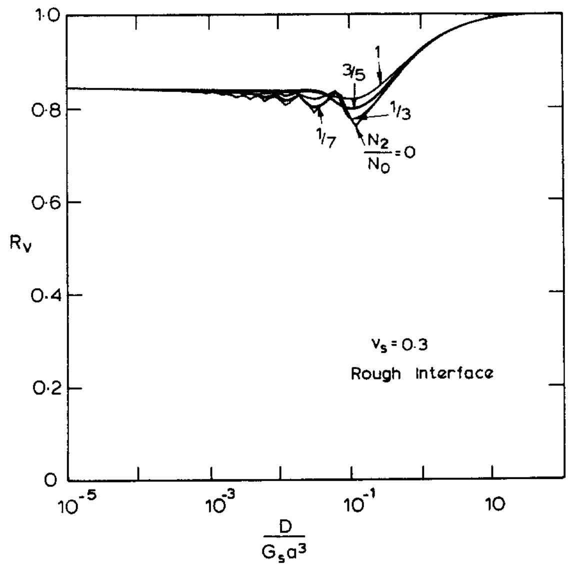

Fig. 5. Influence of hoop force nonuniformity on correction factor for Poisson's ratio.

Höeg[6] and Einstein and Schwartz[7]) to determine the distribution of stress resultants should consider both the influence of the interface condition and Poisson's ratio of the ground ν_s. The present analysis takes the issue further, by considering the effect of interface condition and ν_s on the elastic stability of the buried tube.

It has been found in studies of the uniform hoop force problem[1, 8] that when the ground is incompressible, $\nu_s = 0.5$ (e.g. for an undrained clay), there is no difference in critical hoop compressions for smooth and rough cylinders. This is also true for the nonuniform hoop compression problem, because the nature of the elastic ground resistance is essentially the same. For this reason, the values of influence coefficient I, Fig. 2, are provided for $\nu_s = 0.5$, and are independent of interface condition.

Correction factors R_ν for $\nu_s \neq 0.5$ are provided in Figs. 3 and 4 for the rough and smooth interface conditions, respectively, and for hoop force nonuniformity $N_2/N_0 = \frac{1}{3}$. The influence of N_2/N_0 on R_ν is illustrated in Fig. 5, where curves are given for a range of values of N_2/N_0, with $\nu_s = 0.3$. It is apparent from this figure that the nonuniform component of hoop force acts to couple the modes of the critical deformation and smooth the transition from mode to mode, without seriously affecting R_ν. For this reason, it is reasonable to neglect the influence of N_2/N_0 on R_ν, and use the values of R_ν provided in Figs. 3 and 4 for the full range of N_2/N_0, $0 < N_2/N_0 < 1$.

In general, it can be concluded that the influence of interface condition and Poisson's ratio of the ground on the elastic stability of a tube under nonuniform hoop compression is largely the same as it is when the hoop compressions are uniform.

Effect of shear and extension in the thin cylinder

The parametric solution to the buried cylinder problem which has been presented was based on the assumption that the buckling deformations of the buried cylinder were without extensional and transverse shearing components. The appropriateness of this assumption has been examined elsewhere (see Moore and Booker[15]) by making comparisons between the two solutions which were developed, eqns (15) and (17). It was concluded in that study that, for most practical cases, the extensional and transverse shearing deformations do not significantly lower the elastic stability of buried flexible cylinders (i.e. $R_e = 1$). The possible exception is thicker cylinders $t/a > 0.01$, under highly nonuniform initial stress, but for these cases the elastic solution is probably inappropriate, as the high bending moments will result in an inelastic structural response.

4. SUMMARY AND CONCLUSIONS

General linearised differential equations of equilibrium have been derived, after Herrmann and Armenakas[4], which describe the equilibrium of an initially circular ring under the influence of nonuniform initial stress. More convenient middle surface deformation quantities than those previously adopted have been used to simplify the equations and facilitate their use in solving ring deformation and stability problems. The equations for three alternative ring theories have been presented:

(i) An inextensional theory where only bending deformations are considered.
(ii) An extensional theory, where bending and extensional deformations are considered.
(iii) A more comprehensive theory, which considers bending, extensional, and transverse shearing deformations.

The problem of a flexible cylindrical tube buried in an elastic continuum prestressed by nonhydrostatic biaxial field stresses has been considered. The classical eigenvalue problem has been formulated to determine the distributions of nonuniform stress resultants which result in instability of the buried cylinder in its initial circular configuration (i.e. when the influence of prebuckling deformations on elastic stability is neglected). Details of the appropriate equations have been presented for both the inextensional theory and the more comprehensive theory which considers bending, extensional, and shearing deformations.

The behaviour of buried cylinders under the following conditions was considered:

(i) Two ideal soil-structure interface conditions—perfectly rough and perfectly smooth;
(ii) A range of relative cylinder stiffnesses—from very flexible to very stiff;
(iii) A range of Poisson's ratios of the ground;
(iv) A range of relative cylinder thicknesses;
(v) A variety of distributions of initial stress resultants.

From this examination it was concluded that

(1) The distribution of initial hoop compressions in the cylinder is dominant in determining the cylinder's elastic stability. Shear force and bending moment resultants are relatively unimportant.
(2) Nonuniformity in the distribution of hoop compressions significantly influences the cylinder stability. It influences the form of the critical deformations as well as the level of critical stress.
(3) The influence of interface condition and Poisson's ratio of the ground are largely independent of the nonuniformity of the initial hoop compressions.
(4) The simplified inextensional solution can be used instead of the more comprehensive solution to evaluate the critical hoop forces under most practical circumstances.
(5) It is always conservative when checking the elastic stability of buried cylinders under nonuniform hoop compressions, to use the maximum value of hoop force in conjunction with the solution of the uniform hoop force problem. The practice of using the average hoop compression in conjunction with the uniform hoop force solution (Duns and Butterfield[12], Gumbel and Wilson[13]) is generally unconservative.

A parametric solution based on the inextensional theory has also been presented in the form of influence charts, which may be used directly in hand calculations to predict the elastic stability of deeply buried cylinders under the influence of nonuniform hoop forces.

REFERENCES

1. M. J. Forrestal and G. Herrmann, Buckling of a long cylindrical shell surrounded by an elastic medium. *Int. J. Solids Structures* **1**, 297–309 (1965).
2. J. A. Cheney, Buckling of thin-walled cylindrical shells in soil. *TRRL Supplementary Report 204*, Crowthorne, Victoria (1976).

3. L. D. Baikie and G. G. Meyerhof, Buckling behaviour of buried flexible structures. *Proc. 4th Intl. Conf. on Numerical Methods in Geomechanics*. Edmonton, Canada, 875–882 (1982).
4. G. Herrmann and A. E. Armenakas, Dynamic behaviour of cylindrical shells under initial stress. *Proc. 4th Natl. Congress of Appd. Mech. ASME* **1**, 203–213 (1962).
5. I. Mirsky and G. Herrmann, Nonaxially symmetric motions of cylindrical shells. *J. of Accoustical Soc. of Am.* **29**(10), 1116–1123; **31**(2), 250 (1957, 1959).
6. K. Höeg, Stresses against underground structural cylinders. *J. Soil Mech. Fndns. Divn. ASCE* **94**(SM4), 833–858 (1968).
7. H. H. Einstein and C. W. Schwartz, Simplified analysis for tunnel supports. *J. Geotech. Engng. Divn. ASCE* **105**(CT4), 499–518 (1978).
8. I. D. Moore and J. R. Booker, Simplified theory for the behaviour of buried flexible cylinders under the influence of uniform hoop compression. *Int. J. Solids Structures* **21**, 929–941 (1985).
9. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*. Second Ed., p. 116, McGraw-Hill, New York (1951).
10. I. D. Moore and J. R. Booker, A numerical investigation into the stability of buried flexible tubes. (In press).
11. C. V. Smith and G. J. Simitses, Effect of shear and load behaviour on ring stability. *J. Eng. Mech. Divn. ASCE* **(EM3)**, 559–569 (1969).
12. C. S. Duns and R. Butterfield, Flexible buried cylinders Part III—buckling behaviour. *Int. J. Rock Mech. Min. Sci.* **8**, 613–627 (1971).
13. J. E. Gumbel and J. Wilson, Iterative design of buried flexible pipes—a fresh approach from basic principles. *Ground Eng.* **14**(4), 36–40 (1981).
14. R. H. Anderson and A. P. Boresi, Equilibrium and stability of rings under nonuniformly distributed loads. *Proc. 4th U.S. Natl. Congress of Appd. Mech.* **1**, 459–467 (1962).
15. I. D. Moore and J. R. Booker, The behaviour of buried flexible cylinders under the influence of non-uniform hoop compression. *Univ. Sydney, School Civil Mining Engng. Research Report R464*. (1983).
16. K. Washizu, *Variational Methods in Elasticity and Plasticity*. Second Ed., Pergamon Press (1975).
17. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*. Fourth Edition, Dover, New York (1944).
18. L. H. Donnell, *Beams, Plates and Shells*. McGraw-Hill, New York (1976).

APPENDIX: GENERAL DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR A THIN RING

In this appendix, we will develop the equilibrium equations of the long thin circular shell deforming under plane strain conditions, Fig. 1. The derivation parallels that of Herrmann and Armenakas[4], although, in this presentation, it proves more convenient to adopt the midsurface extension and rotation as fundamental variables.

The equations of equilibrium are developed from the equations of virtual work in the deformed state, Washizu[16]

$$\iiint_{V_0} \sigma_{ij}\delta E_{ij}(a + \zeta)\, d\theta\, d\zeta\, d\chi = \iint_S f_j \delta u_j\, dS \tag{A1}$$

where χ, θ, and ζ are the axial, circumferential, and thickness coordinates of the shell, respectively, a is the initial radius of the shell midsurface, σ_{ij} is the second Kirchoff-Piola stress tensor, and E_{ij} is the finite Lagrangian strain tensor within the initial volume V_0. The vector of tractions f_j is applied to the cylinder surface S, and u_j is the vector of displacements. All subscripts i and j range through the index set (χ, θ, ζ). The equation of virtual work for finite deformations (A1) can be developed from the equations of equilibrium of a deformed body. The equation is defined in terms of the arbitrary or "virtual" displacement field δu_j and the strain field variation δE_{ij}, which can be obtained from E_{ij} using the usual conventions of variational calculus.

The finite strain tensor E_{ij} for conditions of plane strain is given by, Love[17]:

$$\begin{aligned} E_{\theta\theta} &= \frac{1}{a+\zeta}\left(u_\zeta + \frac{\partial u_\theta}{\partial\theta}\right) + \frac{1}{2(a+\zeta)^2}\left[\left(u_\zeta + \frac{\partial u_\theta}{\partial\theta}\right) + \left(\frac{\partial u_\zeta}{\partial\theta} - u_\theta\right)^2\right] \\ E_{\zeta\zeta} &= \frac{\partial u_\zeta}{\partial\zeta} + \frac{1}{2}\left[\left(\frac{\partial u_\theta}{\partial\zeta}\right)^2 + \left(\frac{\partial u_\zeta}{\partial\zeta}\right)^2\right] \\ 2E_{\theta\zeta} &= \frac{\partial u_\theta}{\partial\zeta} + \frac{1}{a+\zeta}\left(\frac{\partial u_\zeta}{\partial\theta} - u_\theta\right) + \frac{1}{a+\zeta}\left[\left(u_\zeta + \frac{\partial u_\theta}{\partial\theta}\right)\frac{\partial u_\theta}{\partial\zeta} + \left(\frac{\partial u_\zeta}{\partial\theta} - u_\theta\right)\frac{\partial u_\zeta}{\partial\zeta}\right]. \end{aligned} \tag{A2}$$

Alternative shell theories can be developed on the basis of simplified expressions for finite strain E_{ij}, but the complete virtual work derivation is favoured by the present authors. Since the ring is thin, it is reasonable to assume linear variations of displacement across the shell, viz:

$$\begin{aligned} u_\theta(\theta,\zeta) &= v(\theta) + \zeta\psi_\theta(\theta) \\ u_\zeta(\theta,\zeta) &= w(\theta) + \zeta\psi_\zeta(\theta). \end{aligned} \tag{A3}$$

Stress resultants are defined by the relations

$$
\begin{aligned}
N &= \int_{-t/2}^{t/2} \sigma_{\theta\theta}\, d\zeta \\
M &= \int_{-t/2}^{t/2} \sigma_{\theta\theta}\zeta\, d\zeta \\
Q &= \int_{-t/2}^{t/2} \sigma_{\theta\zeta}\, d\zeta \\
N_{\zeta\zeta} &= \int_{-t/2}^{t/2} \sigma_{\zeta\zeta}(1 + \zeta/a)\, d\zeta .
\end{aligned}
\tag{A4}
$$

The net tractions per unit length of midsurface are defined by the difference of values at top and bottom surfaces:

$$
\begin{aligned}
\tau &= [f_\theta(1 + \zeta/a)]_{-t/2}^{+t/2} \qquad & \sigma &= [f_\zeta(1 + \zeta/a)]_{-t/2}^{+t/2} \\
m_\theta &= [f_\theta\zeta(1 + \zeta/a)]_{-t/2}^{+t/2} \qquad & m\zeta &= [f_\zeta\zeta(1 + \zeta/a)]_{-t/2}^{+t/2}.
\end{aligned}
\tag{A5}
$$

Equations (A1, A2, A3, A4, A5) lead to the following set of equilibrium equations:

$$
\begin{aligned}
N + N\epsilon - \frac{M\epsilon}{a} + \frac{M}{a}\frac{\partial\psi_\theta}{\partial\theta} + \frac{M}{a}\psi_\zeta + Q\psi_\theta - F_\epsilon &= 0 \\
N\alpha - \frac{M\alpha}{a} - \frac{M\psi_\theta}{a} + \frac{M}{a}\frac{\partial\psi_\zeta}{\partial\theta} + Q + Q\psi_\zeta - F_\alpha &= 0 \\
-\frac{1}{a}\frac{\partial M}{\partial\theta} - \frac{\partial}{\partial\theta}\left(\frac{M\epsilon}{a}\right) - \frac{M\alpha}{a} + Q + Q\epsilon + N_{\zeta\zeta}\psi_\theta - \frac{m_\theta}{a} &= 0 \\
\frac{M}{a} + \frac{M\epsilon}{a} - \frac{\partial}{\partial\theta}\left(\frac{M\alpha}{a}\right) + Q\alpha + N_{\zeta\zeta}(1 + \psi_\zeta) - \frac{m_\zeta}{a} &= 0,
\end{aligned}
\tag{A6}
$$

where the extension ϵ and rotation α of the midsurface can be expressed in terms of the midsurface displacements

$$
\begin{aligned}
\epsilon &= \frac{1}{a}\left(\frac{\partial v}{\partial\theta} + w\right) \\
\alpha &= \frac{1}{a}\left(\frac{\partial w}{\partial\theta} - v\right)
\end{aligned}
\tag{A7}
$$

and the "tractions" F_ϵ and F_α are defined by:

$$
\begin{aligned}
\sigma &= -\frac{\partial F_\alpha}{\partial\theta} + F_\epsilon \\
\tau &= -\frac{\partial F_\epsilon}{\partial\theta} - F_\alpha .
\end{aligned}
\tag{A8}
$$

Unique solutions of eqn (A8) for F_α and F_ϵ exist, because the general solutions

$$
\begin{aligned}
F_\alpha &= C_1 \cos\theta + C_2 \sin\theta \\
F_\epsilon &= C_2 \cos\theta - C_1 \sin\theta
\end{aligned}
$$

lead to nonperiodic deformations such as $w = C_3\theta\cos\theta$, which must be zero (i.e. $C_1 = C_2 = 0$).

The eqns (A6) are valid for any general set of finite deformations ϵ, α, ψ_θ, ψ_ζ with their associated stress resultants N, M, Q, $N_{\zeta\zeta}$ and tractions F_ϵ, F_α, m_θ, m_ζ. We now restrict the discussion to small deformations ϵ, α, ψ_θ, ψ_ζ, so that a linear theory can be developed where nonlinear deformation terms are neglected. On this basis, we assume Hookean relations between components of second Kirchhoff-Piola stress and linear components of strain. Integration through the shell thickness then leads to expressions for the stress resultants

$$
\begin{aligned}
N &= \left(H + \frac{D}{a^2}\right)\epsilon - \frac{D}{a^2}\frac{\partial\psi_\theta}{\partial\theta} \\
M &= \frac{D}{a}\frac{\partial\psi_\theta}{\partial\theta} - \frac{D}{a}\epsilon \\
Q &= \kappa G_l t(\psi_\theta + \alpha),
\end{aligned}
\tag{A9}
$$

where it is assumed that normal stress $\sigma_{\zeta\zeta}$ is negligible compared with the membrane stress $\sigma_{\theta\theta}$ (i.e. conditions of plane stress apply). This is the usual approximation employed when developing thin shell theories (see, for example, Donnell[18]), and the two straightforward examples which follow demonstrate its validity. For a ring loaded with uniform external pressure p, $\sigma_{\theta\theta} = pa/t$, while the normal stress $\sigma_{\zeta\zeta}$ ranges from zero to

p. When the applied loads are nonuniform, the bending moment M induces a normal stress $\sigma_{\zeta\zeta}$ ranging from 0 to $\sigma M/at$, while the same moment induces circumferential stress $\sigma_{\theta\theta}$ which varies between $\pm\sigma M/t^2$.

Before substituting the expressions for stress resultants (A9), the equations of equilibrium are further simplified. Firstly, the stress resultant $N_{\zeta\zeta}$ is eliminated from the third equation of (A6), using the fourth of (A6). Then ψ_ζ and m_ζ are discarded, because they are negligible compared to the other deformations ϵ, α, ψ_θ and "tractions" F_ϵ, F_α, and m_θ, respectively. (The term ψ_ζ was originally included in eqn (A3) so that the fourth equation of (A6) could be formulated.) The equations outlined in Section 2 then follow on sequential application of the special conditions

$$\psi_\theta + \alpha = 0 \tag{A10a}$$

and

$$\epsilon = 0, \tag{A10b}$$

and substitution of the constitutive equations (A9).